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REJECTION METHODS FOR SAMPLING
FROM THE NORMAL DISTRIBUTION

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ABSTRACT

A general rejection method is presented which can be applied to generate random variates from any continuous probability distribution. This general method is used to derive three algorithms for generating normal random variates.

1. Introduction

Over the past twenty years, Monte Carlo simulation methods have been increasingly employed to solve diverse problems in the engineering, management and physical sciences. This popularity is in part attributable to the method's relative simplicity compared to analytical techniques. Moreover, Monte Carlo simulation methods can be used to investigate numerous variants of the original model without the corresponding experimental costs. The ease with which the investigator exercises his model is related to the efficiency of generating variates or sample data from his assumed stochastic model.

One of the most widely used components in stochastic models is the normal probability distribution. Hence, it is desirable to have a convenient method for sampling from this distribution. Among the more successful algorithms developed for generating normal variates include (1) the Box-Muller transformation [3], (2) Marsaglia's fast procedure [6], (3) the trapezoidal method given by Ahrens and Dieter [1, 2], and (4) the composition method of Kinderman and Ramage [5]. These methods vary considerably in complexity and operating performance. The easiest procedure to implement but the slowest to execute is the Box-Muller transformation method. Conversely, Marsaglia's procedure is exceedingly fast but requires considerable core storage. In this paper three additional methods are derived for generating normal variates. In Section 2

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the general rejection method which is the basis of these algorithms is derived. The three new methods are presented in Section 3.

2. A General Rejection Method

In this section we propose a general rejection method which extends Von Neumann's original rejection method [8]. Let $f(x)$ be the probability density function (pdf) with support Ω from which variates are to be generated. Let $h(x;\theta)$ be another pdf (where θ is a "free" parameter to be determined) satisfying $f(x) \leq \alpha\, h(x;\theta)$, for all $x \in \Omega$ and $\alpha \geq 1$. The general rejection method is as follows:

1. Generate a random variate x from $h(x;\theta)$.

2. Generate a uniform (0,1) random variate u.

3. If $u \leq T(x) = f(x)/\alpha h(x;\theta)$, accept $X = x$ as a random variate from $f(x)$. Otherwise, return to 1.

In the above procedure, $1/\alpha$ is designated the efficiency of the rejection method. A justification for this procedure has been given by one of the authors [4] and is summarized below.

Using Bayes' Theorem, we have

$$P[X = x \mid U \leq f(x)/\alpha h(x;\theta)] = \frac{P[U \leq f(X)/\alpha h(X;\theta) \mid X = x]\; h(x;\theta)}{P[U \leq f(x)/\alpha h(x;\theta)]}$$

We can directly compute the following:

$$P[U < f(X)/\alpha h(X;\theta) \mid X = x] = f(x)/\alpha h(x;\theta).$$

$$\begin{aligned} P[U < f(x)/\alpha h(x;\theta)] &= \int P[U < f(x)/\alpha h(x;\theta)] \cdot h(x;\theta)dx \\ &= \int f(x)/\alpha\, dx \\ &= 1/\alpha. \end{aligned}$$

Substituting yields

$$\begin{aligned} P[X = x \mid U \leq f(x)/\alpha h(x;\theta)] &= \{f(x)/\alpha h(x;\theta)\} \cdot h(x;\theta)/(1/\alpha) \\ &= f(x)\ . \end{aligned}$$

The selection of $h(x;\theta)$ governs the eventual success of the rejection method. The following criteria should be used in selecting $h(x;\theta)$:

1. The algorithm used for generating random variates from $h(x;\theta)$ must be efficient.

2. The efficiency of the procedure ($\epsilon = 1/\alpha$) must be large, which occurs if α is close to 1 or equivalently, if $h(x;\theta)$ is similar in shape to $f(x)$.

The parameter θ is computed to maximize the efficiency. Define $L(\theta) = \max_{x\epsilon\Omega} f(x)/h(x;\theta)$, and determine $\theta = \theta^*$ for which $L(\theta)$ attains a minimum. The optimal efficiency is $1/\alpha^* = 1/L(\theta^*)$. In the next section we illustrate this computation for the normal density f and three "similar" densities $h(x;\theta)$.

3. Generating Standard Normal Variates

The standard normal pdf is given by

$$f(x) = (1/\sqrt{2\pi}) \cdot \exp(-x^2/2), \quad -\infty < x < \infty. \tag{3.1}$$

In this section three methods for sampling from (3.1) are given. The methods are based on the general rejection method presented in Section 2 but using different choices of $h(x;\theta)$. The proposed methods are presented in algorithmic form below. A uniform (0,1) generator is assumed to be available

3.1 Method 1: Exponential

Since the standard normal distribution is symmetrical about zero, we can consider the distribution of $Y = |X|$ having pdf

$$g(y) = \sqrt{2/\pi} \cdot \exp\,(-y^2/2), \qquad y \geq 0. \tag{3.2}$$

After obtaining samples from (3.2), a sign (+ or -) is randomly assigned to each sample to obtain normal variates.

To obtain samples from (3.2), the exponential density given by

$$h(x;\theta) = \frac{1}{\theta} \exp\,(-x/\theta), \qquad x,\ \theta > 0,$$

is employed. For this choice of h, θ^* can be computed directly. First

$$L(\theta) = \max_{x\epsilon\Omega} \sqrt{2/\pi}\ \theta \exp\,(x/\theta - x^2/2).$$

The right hand side is maximized for $x = 1/\theta$, so that $L(\theta) = \sqrt{2/\pi}\ \theta \exp\,(1/2\theta^2)$. Next, the problem $\min_{\theta>0} L(\theta) = \min_{\theta>0} \sqrt{2/\pi}\ \theta \exp\,(1/2\theta^2)$ is solved, yielding $\theta^* = 1$. Moreover, the optimal efficiency is $1/\alpha^* = 0.7605$ and $T(x) = \exp\,(-x^2 + x - 0.5)$. The resulting algorithm is as follows:

1. Generate an exponential variate z from a uniform variate u via $z = -\ln(u)$.

2. Generate a uniform variate v. If $v > T(z)$, reject z and return to Step 1.

3. Generate a uniform variate w. If $w \geq 0.5$, accept the variate $x = -z$.

4. Accept the variate $x = z$.

The computational efficiency (CPU time) of the procedure can be improved by generating the exponential variate in Step 1 by faster but more complex methods [1,7].

3.2 Method 2: Logistic

In this case the distribution of interest is the standard normal density (3.1), and $h(x;\theta)$ is the logistic pdf:

$$h(x;\theta) = \frac{\exp(-x/\theta)}{\theta \cdot [1 + \exp(-x/\theta)]^2}, \quad -\infty < x < \infty.$$

It can be shown numerically that $\theta^* = 0.626657$, $1/\alpha^* = 0.9196$, and $T(x) = 0.25\,[1 + \exp(-1.5957\,x)]^2 \cdot \exp(-x^2/2 + 1.5957\,x)$.

The resulting algorithm is as follows:

1. Generate a logistic variate z from a uniform variate u via $z = -0.626657\,\ln(1/u - 1)$.

2. Generate a uniform variate v. If $v > T(z)$, reject z and return to Step 1.

3. Accept $x = z$.

The optimal efficiency of this algorithm is improved over the previous method. However, the evaluation of the function T is more tedious. Simple upper and lower bounds (perhaps linear functions) for T could further improve this method.

3.3 Method 3: Cauchy

Finally, the Cauchy pdf is considered:

$$h(x;\theta) = \left(\pi\theta\right)^{-1} \left(1 + (x/\theta)^2\right)^{-1}, \qquad -\infty < x < \infty.$$

It is easy to verify that $\theta^* = 1$, $1/\alpha^* = 0.6578$, and $T(x) = 0.82436\,(1 + x^2)\exp(-x^2/2)$. The corresponding algorithm is as follows:

1. Generate a Cauchy variate z from a uniform variate u via $z = \tan \pi(u - 0.5)$.

2. Generate a uniform variate v. If $v > T(z)$, reject z and return to Step 1.

3. Accept $x = z$.

The computational efficiency of this procedure can be improved. The probability that the absolute value of a standard normal variate, $|x|$, will be greater than 4.0 is only 0.000064. If we can neglect this probability, which implies we are sampling from the truncated normal distribution with density

$$f(x) = \frac{1}{\sqrt{2\pi}\,(0.999936)} \exp(-x^2/2), \; -4.0 \le x \le 4.0,$$

we can add the following step to the above algorithm:

0: Generate a uniform random variate v and set $u = 0.844\,v + 0.078$.

Summary

A general rejection method has been presented which can be used to generate variates from arbitrary continuous probability distributions. The method was used to derive three algorithms for generating normal variates. These algorithms are based upon maximizing the efficiency with respect to the parameters in the exponential, logistic and Cauchy distributions, respectively.

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